

The noncommutative family Atiyah-Patodi-Singer index theorem

Yong Wang

Abstract

In this paper, we define the eta cochain form and prove its regularity when the kernel of a family of Dirac operators is a vector bundle. We decompose the eta form as a pairing of the eta cochain form with the Chern character of an idempotent matrix and we also decompose the Chern character of the index bundle for a fibration with boundary as a pairing of the family Chern-Connes character for a manifold with boundary with the Chern character of an idempotent matrix. We define the family b -Chern-Connes character and then we prove that it is entire and give its variation formula. By this variation formula, we prove another noncommutative family Atiyah-Patodi-Singer index theorem. Thus, we extend the results of Gezler and Wu to the family case.

Keywords: Eta cochain form; family Chern-Connes character for manifolds with boundary; family b -Chern-Connes character; variation formula.

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1 Introduction

In [APS], Atiyah-Patodi-Singer introduced the eta invariant and proved their famous Atiyah-Patodi-Singer index theorem for manifolds with boundary. In [BC], using Cheeger's cone method, Bismut and Cheeger defined the eta form which is a family version of the eta invariant and extended the APS index formula to the family case under the condition that all boundary Dirac operators are invertible. In [MP1,2], using the Melrose's b -calculus, Melrose and Piazza extended the Bismut-Cheeger family index theorem to the case that boundary Dirac operators are not invertible. In [Do], Donnelly extended the APS index theorem to the equivariant case by modifying the Atiyah-Patodi-Singer original method. In [Zh], Zhang got this equivariant Atiyah-Patodi-Singer index theorem by using a direct geometric method in [LYZ].

On the other hand, in [Wu], Wu proved the Atiyah-Patodi-Singer index theorem in the framework of noncommutative geometry. To do so, he introduced the eta cochain (called the higher eta invariant in [Wu]) which is a generalization of the classical Atiyah-Patodi-Singer eta invariant in [APS], then proved its regularity by using the Getzler symbol calculus [Ge1] as adopted in [BF] and computed its radius of convergence. Subsequently, he proved the variation formula of eta cochains, using which he

got the noncommutative Atiyah-Patodi-Singer index theorem. In [Ge2], using super-connection, Getzler gave another proof of the noncommutative Atiyah-Patodi-Singer index theorem, which was more difficult, but avoided mention of the operators b and B of cyclic cohomology. In [Wa1], we defined the equivariant eta cochain and proved its regularity using the method in [CH], [Fe] and [Zh]. Then we proved an equivariant noncommutative Atiyah-Patodi-Singer index theorem. In [Wa2], we defined infinitesimal equivariant eta cochains and proved their regularity. In [LMP], Lesch, Moscovici and Pflaum presented the Chern-Connes character of the Dirac operator associated to a b -metric on a manifold with boundary in terms of a retracted cocycle in relative cyclic cohomology. Blowing-up the metric one recovered the pair of characteristic currents that represent the corresponding de Rham relative homology class, while the blowdown yielded a relative cocycle whose expression involves higher eta cochains and their b -analogues. The corresponding pairing formula with relative K -theory classes captured information about the boundary and allowed to derive geometric consequences. In [Xi], Xie proved an analogue for odd dimensional manifolds with boundary, in the b -calculus setting, of the higher Atiyah-Patodi-Singer index theorem by Getzler and by Wu. Xie also obtained a natural counterpart of the eta invariant for even dimensional closed manifolds.

The purpose of this paper is to extend the theorems due to Getzler and Wu to the family case. our main theorems are as follows (for related definitions, see Sections 3-5).

Theorem 3.6 *Suppose that all $D_{M,z}$ are invertible with λ the smallest positive eigenvalue of all $|D_{M,z}|$. We assume that $\|d(p|_M)\| < \lambda$ and $p \in M_{r \times r}(C_*^\infty(N))$, then in the cohomology of X*

$$\text{ch}[\text{Ind}(pD_{z,+,\varepsilon}p)] = \langle \tau(B), \text{Ch}(p) \rangle. \quad (1.1)$$

Theorem 5.1 *Suppose that all $D_{M,z}$ are invertible with λ the smallest positive eigenvalue of all $|D_{M,z}|$. We assume that $\|d(p|_M)\| < \lambda$ and $p \in M_{r \times r}(C_{\text{exp}}^\infty(\hat{N}))$, then in the cohomology of X*

$$\text{ch}[\text{Ind}(pD_{z,+}p)] = \int_{\hat{N}/X}^b \hat{A}(R^{\hat{N}/X}) \text{ch}(\text{Imp}) - \langle \hat{\eta}^*(B^M), \text{ch}_*(p_M) \rangle. \quad (1.2)$$

The above theorems obviously apply with no mayor modifications to the twisted case by an extra odd differential form. This paper is organized as follows: in Section 2, we define the eta cochain form and prove its regularity when the kernel of a family of Dirac operators is a vector bundle.. In Section 3, we decompose the eta form as a pairing of the eta cochain form with the Chern character of an idempotent matrix and we also decompose the Chern character of the index bundle for a fibration with boundary as a pairing of the family Chern-Connes character for manifolds with boundary with the Chern character of an idempotent matrix. In Section 4, We define the family b -Chern-Connes character and then we prove that it is entire and give its variation formula. In Section 5, by this variation formula, we prove another noncommutative family Atiyah-Patodi-Singer index theorem. Thus, we extend the

results of Gezler and Wu to the family case.

2 The eta cochain form

In this Section, we define the eta cochain form and prove its regularity.

Firstly, we recall the Bismut superconnection. Let M be a $n + q$ dimensional compact connected manifold without boundary and X be a q dimensional compact connected manifold without boundary. We assume that $\pi : M \rightarrow X$ is a submersion of M onto X , which defines a fibration of M with the fibre Z . For $y \in X$, $\pi^{-1}(y)$ is a submanifold M_y of M . Denote by TZ the n -dimensional vector bundle on M whose fibre $T_x M_{\pi(x)}$ is the tangent space at x to the fibre $M_{\pi(x)}$. We assume that M and X are oriented. We take a smooth horizontal subbundle $T^H M$ of TM . A vector field $X \in \Gamma(X, TX)$ will be identified with its horizontal lift $X^H \in \Gamma(M, T^H M)$. Moreover $T_x^H M$ is isomorphic to $T_{\pi(x)} X$ via π_* . We take a Riemannian metric on X and then lift the Euclidean scalar product g_X of TX to $T^H M$. We further assume that TZ is endowed with a scalar product g_Z . Thus we can introduce on TM a new scalar product $g_X \oplus g_Z$, and denote by ∇^L the Levi-Civita connection on TM with respect to this metric. Set ∇^X denote the Levi-Civita connection on TX and we still denote by ∇^X the pullback connection on $T^H M$. Let $\nabla^Z = P_Z(\nabla^L)$ where P_Z denotes the orthogonal projection to TZ . Set $\nabla^\oplus = \nabla^X \oplus \nabla^Z$ and $S = \nabla^L - \nabla^\oplus$ and T be the torsion tensor of ∇^\oplus . Denote by $SO(TZ)$ the $SO(n)$ bundle of oriented orthonormal frames in TZ . Now we assume that the bundle TZ is spin. Denote by $S(TZ)$ the associated spinor bundle and ∇^Z can be lifted to a connection on $S(TZ)$. Let D be the Dirac operator in the tangent direction defined by $D = \sum_{j=1}^n c(e_j^*) \nabla_{e_j}^{S(TZ)}$ where $\nabla^{S(TZ)}$ is a spin connection on $S(TZ)$. Set E be the vector bundle $\pi^*(\wedge T^* X) \otimes S(TZ)$. Then the Bismut superconnection acting on E is defined by

$$B = D + \sum_{\alpha=1}^q f_\alpha^* \wedge (\nabla_{f_\alpha}^{S(TZ)} + \frac{1}{2} k(f_\alpha)) - \frac{1}{4} c(T), \quad (2.1)$$

where

$$k(f_\alpha) = \sum_{j=1}^n \left\langle \nabla_{f_\alpha}^{TZ} e_j - [f_\alpha, e_j], e^j \right\rangle, \quad c(T) = - \sum_{\alpha < \beta} \sum_j f^\alpha \wedge f^\beta c(e_j) \left\langle [f_\alpha^H, f_\beta^H], e_j \right\rangle. \quad (2.2)$$

Let $\psi_t : dy_\alpha \rightarrow \frac{dy_\alpha}{\sqrt{t}}$ be the rescaling operator. Let $B_t = \sqrt{t} \psi_t(B)$ and $F_t = B_t^2$. Let tr^{even} denote taking the trace with value in $\Omega^{\text{even}}(X)$. When $\dim Z$ is odd, for $a_0, \dots, a_{2k} \in C^\infty(M)$, we define the family cochain $\text{ch}_{2k}(B_t, \frac{dB_t}{dt})$ by the formula:

$$\begin{aligned} & \text{ch}_{2k}(B_t, \frac{dB_t}{dt})(a_0, \dots, a_{2k}) \\ &= \sum_{j=0}^{2k} (-1)^j \langle a_0, [B_t, a_1], \dots, [B_t, a_j], \frac{dB_t}{dt}, [B_t, a_{j+1}], \dots, [B_t, a_{2k}] \rangle_t, \end{aligned} \quad (2.3)$$

If A_j ($0 \leq j \leq q$) are operators on $\Gamma(E)$, we define:

$$\langle A_0, \dots, A_q \rangle_t = \int_{\Delta_q} \text{tr}^{\text{even}}[A_0 e^{-\sigma_0 F_t} A_1 e^{-\sigma_1 F_t} \dots A_q e^{-\sigma_q F_t}] d\sigma, \quad (2.4)$$

where $\Delta_q = \{(\sigma_0, \dots, \sigma_q) | \sigma_0 + \dots + \sigma_q = 1, \sigma_j \geq 0\}$ is a simplex in \mathbf{R}^q . When $\dim Z$ is even, in (2.4), we use str instead of tr^{even} and define $\text{ch}_{2k}(B_t, \frac{dB_t}{dt})$.

We assume that the kernel of D is a complex vector bundle. Formally, the **eta cochain form** is defined to be an even cochain sequence by the formula:

$$\hat{\eta}_{2k}(B) = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{ch}_{2k}(B_t, \frac{dB_t}{dt}) dt, \quad \text{when } \dim Z \text{ is odd}; \quad (2.5)$$

$$\hat{\eta}_{2k}(B) = \int_0^\infty \text{ch}_{2k}(B_t, \frac{dB_t}{dt}) dt, \quad \text{when } \dim Z \text{ is even.} \quad (2.6)$$

This integral makes sense by the following Lemma 2.1 and Lemma 2.5. Then $\hat{\eta}_0(B)(1)$ is the eta form defined by Bismut and Cheeger in [BC]. In order to prove that the above definition is well defined, it is necessary to check the integrality near the two ends of the integration. Firstly, the regularity at infinity comes from the following lemma.

Lemma 2.1 *We assume that the kernel of D is a complex vector bundle. For $a_0, \dots, a_{2k} \in C^\infty(M)$, we have*

$$\text{ch}_{2k}(B_t, \frac{dB_t}{dt})(a_0, \dots, a_{2k}) = O(t^{-\frac{3}{2}}), \quad \text{as } t \rightarrow \infty. \quad (2.7)$$

Proof. Since the kernel of D is a complex vector bundle, our proof is very similar to the proof of Lemma 3.5 in [Wa2] (see revised version arXiv:1307.8189). We just use Lemma 9.4 in [BGV] instead of Lemma 3.4 in [Wa2]. We use $D + \frac{c(T)}{4}$ and $\psi_t : dy_\alpha \rightarrow \frac{dy_\alpha}{\sqrt{t}}$ instead of $D - \frac{c(\hat{X})}{4}$ and $\psi_t : \hat{X} \rightarrow \frac{\hat{X}}{t}$ in Lemma 3.5 in [Wa2] respectively where \hat{X} is the Killing vector field. We note that $\text{ch}_{2k}(B_t, \frac{dB_t}{dt})$ corresponds to $\frac{1}{2\sqrt{t}} \text{ch}_k(\sqrt{t} D_{-\hat{X}}, D_{\hat{X}})$ in [Wa2]. Comparing with the single operator case in Lemma 2 in [CM], the operator $[B_t, a_j] = \sqrt{t}[c(d_Z a_j) + \frac{1}{\sqrt{t}} d_X a_j \wedge]$ is instead of $\sqrt{t}[D, a_j]$ and $\delta_t(g)$ in Lemma 9.21 in [BGV] emerges, where $\delta_t(g) = 1 + O(t^{-\frac{1}{2}})S_0$ and S_0 is a smooth operator. By these differences, in the discussions of Lemma 2 in [CM], the number of copies of $e^{-\sigma_t D^2}(I - H)$ may be less than $\frac{k}{2} + 1$. But the coefficients of S_0 and $d_X a_j \wedge$ are $O(t^{-\frac{1}{2}})$. Through careful observations, we still get (2.7). \square

In the following, we prove the regularity at zero of the eta cochain form. We know that $\frac{dB_t}{dt} = \frac{1}{2\sqrt{t}} \psi_t(D + \frac{c(T)}{4})$. We introduce the Grassmann variable dt which anticommutes with $c(e_j)$ and dy_α . Set $\hat{F} = F + dt(D + \frac{c(T)}{4})$. Let

$$\text{ch}_{2k}(\hat{F})(a_0, \dots, a_{2k}) = t^k \int_{\Delta_{2k}} \psi_t \text{tr}^{\text{even}}[a_0 e^{-t\sigma_0 \hat{F}} [B, a_1] \dots [B, a_{2k}] e^{-t\sigma_{2k} \hat{F}}] d\sigma. \quad (2.8)$$

By the Duhamel principle and $(dt)^2 = 0$, we have

$$e^{-t\sigma_j\hat{F}} = e^{-t\sigma_j F} - t dt \int_0^{\sigma_j} e^{-t(\sigma_j - \xi)F} (D + \frac{c(T)}{4}) e^{-t\xi F} d\xi. \quad (2.9)$$

By (2.8) and (2.9), we have

$$\text{ch}_{2k}(\hat{F})(a_0, \dots, a_{2k}) = \text{ch}_{2k}(F)(a_0, \dots, a_{2k}) + t^{\frac{3}{2}} \text{ch}_{2k}(B_t, \frac{dB_t}{dt})(a_0, \dots, a_{2k}) dt. \quad (2.10)$$

Let A be an operator and l be a positive interger. Write

$$A^{[l]} = [\hat{F}, A^{[l-1]}], \quad A^{[0]} = A, \quad A^{(l)} = [F, A^{(l-1)}], \quad A^{(0)} = A.$$

Similar to Lemma 4.4 in [Wa3], we have

Lemma 2.2 *Let A a finite order fibrewise differential operator with form coefficients, then for any $s > 0$, we have:*

$$e^{-sF} A = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l A^{(l)} e^{-sF} + (-1)^N s^N A^{(N)}(s); \quad (2.11)$$

$$e^{-s\hat{F}} A = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l A^{[l]} e^{-s\hat{F}} + (-1)^N s^N A^{[N]}(s), \quad (2.12)$$

where $A^{(N)}(s)$ and $A^{[N]}(s)$ are given by

$$A^{(N)}(s) = \int_{\Delta_N} e^{-u_1 s F} A^{(N)} e^{-(1-u_1)sF} du_1 du_2 \dots du_N; \quad (2.13)$$

$$A^{[N]}(s) = \int_{\Delta_N} e^{-u_1 s \hat{F}} A^{[N]} e^{-(1-u_1)s\hat{F}} du_1 du_2 \dots du_N. \quad (2.14)$$

As in [CH], [Fe], [Wa3], by Lemma 2.2, we have for a sufficient large N ,

$$\begin{aligned} & \text{ch}_{2k}(F)(a_0, \dots, a_{2k}) \\ &= \psi_t \sum_{\lambda_1, \dots, \lambda_{2k}=0}^N \frac{(-1)^{\lambda_1 + \dots + \lambda_{2k}}}{\lambda_1! \dots \lambda_{2k}!} C t^{|\lambda|+k} \text{tr}^{\text{even}} \left[a_0[B, a_1]^{(\lambda_1)} \dots [B, a_{2k}]^{(\lambda_{2k})} e^{-tF} \right] + O(t^{\frac{3}{2}}); \end{aligned} \quad (2.15)$$

$$\begin{aligned} & \text{ch}_{2k}(\hat{F})(a_0, \dots, a_{2k}) \\ &= \psi_t \sum_{\lambda_1, \dots, \lambda_{2k}=0}^N \frac{(-1)^{\lambda_1 + \dots + \lambda_{2k}}}{\lambda_1! \dots \lambda_{2k}!} C t^{|\lambda|+k} \text{tr}^{\text{even}} \left[a_0[B, a_1]^{[\lambda_1]} \dots [B, a_{2k}]^{[\lambda_{2k}]} e^{-t\hat{F}} \right] + O(t^{\frac{3}{2}}), \end{aligned} \quad (2.16)$$

where C is a constant. Recall Lemma 2.17 in [Wa3] which extends the corresponding Lemma in [Po] and [PW]

Let U be an open subset of \mathbf{R}^n . We define Volterra symbols and Volterra ΨDO s

on $U \times \mathbf{R}^{n+1}/0$ as follows.

Definition 2.3 The set $S_V^m(U \times \mathbf{R}^{n+1}) \otimes \wedge T_z^* B$, $m \in \mathbf{Z}$, consists of smooth functions $q(x, \xi, \tau)$ on $U \times \mathbf{R}^n \times \mathbf{R}$ with an asymptotic expansion $q \sim \sum_{j \geq 0} q_{m-j}$, where:
 $-q_l \in C^\infty(U \times [(\mathbf{R}^n \times \mathbf{R}) \setminus 0]) \otimes \wedge T_z^* B$ is a homogeneous Volterra symbol of degree l , i.e. q_l is parabolic homogeneous of degree l and satisfies the property

- (i) q extends to a continuous function on $(\mathbf{R}^n \times \overline{\mathbf{C}_-}) \setminus 0$ in such way to be holomorphic in the last variable when the latter is restricted to \mathbf{C}_- .
- The sign \sim means that, for any integer N and any compact $K \subset U$, there is a constant $C_{NK\alpha\beta k} > 0$ such that for $x \in K$ and for $|\xi| + |\tau|^{\frac{1}{2}} > 1$ we have

$$\|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j < N} q_{m-j})(x, \xi, \tau)\| \leq C_{NK\alpha\beta k} (|\xi| + |\tau|^{\frac{1}{2}})^{m-N-|\beta|-2k}. \quad (2.17)$$

For $q = \sum_l q_l \omega^l$ where $q_l \in S_V^m(U \times \mathbf{R}^{n+1})$ and $\omega^l \in \wedge^l T_z^* B$, we define $\|q\| = \sum_l |q_l| \|\omega^l\|$ and $\|\omega^l\|$ is the norm of ω^l in $(\wedge^l T_z^* B, g_z^{TB})$.

Definition 2.4 The set $\Psi_V^m(U \times \mathbf{R}, \wedge T_z^* B)$, $m \in \mathbf{Z}$, consists of continuous operators Q from $C_c^\infty(U_x \times \mathbf{R}_t, \wedge T_z^* B)$ to $C^\infty(U_x \times \mathbf{R}_t, \wedge T_z^* B)$ such that:

- (i) Q has the Volterra property;
- (ii) $Q = q(x, D_x, D_t) + R$ for some symbol q in $S_V^m(U \times \mathbf{R}, \wedge T_z^* B)$ and some smoothing operator R .

In the sequel if Q is a Volterra ΨDO , we let $K_Q(x, y, t - s)$ denote its distribution kernel, so that the distribution $K_Q(x, y, t)$ vanishes for $t < 0$.

Lemma 2.5 (Lemma 2.17 in [Wa3]) Let $Q \in \Psi_V^*(\mathbf{R}^n \times \mathbf{R}, S(T(M_z)) \otimes \wedge^* T_z^* X)$ have Getzler order m and model operator $Q_{(m)}$. Then as $t \rightarrow 0^+$ we have:

- 1) $\sigma[\psi_t K_Q(0, 0, t)]^{(j)} = \omega^{\text{odd}} O(t^{\frac{j-n-m-2}{2}}) + O(t^{\frac{j-n-m-1}{2}})$, if $m - j$ is odd;
 - 2) $\sigma[\psi_t K_Q(0, 0, t)]^{(j)} = t^{\frac{j-n-m-2}{2}} K_{Q_{(m)}}(0, 0, 1)^{(j)} + \omega^{\text{odd}} O(t^{\frac{j-n-m-1}{2}}) + O(t^{\frac{j-n-m}{2}})$, if $m - j$ is even,
- where $[K_Q(0, 0, t)]^{(j)}$ denotes the degree j form component in M_z and $\omega^{\text{odd}} O(t^{\frac{j-n-m-2}{2}})$ denotes that the coefficients of $t^{\frac{j-n-m-2}{2}}$ are in $\wedge^{\text{odd}}(T^* X) \otimes \wedge(T^*(M_z))$.

Lemma 2.6 The following estimate holds

$$\text{ch}_{2k}(B_t, \frac{dB_t}{dt}) \sim O(1) \text{ when } t \rightarrow 0, \quad (2.18)$$

Proof. By (2.10), (2.15) and (2.16), in order to prove Lemma 2.6, we only prove

$$\psi_t t^{|\lambda|+k} \text{tr}^{\text{even}}[a_0[B, a_1]^{[\lambda_1]} \dots [B, a_{2k}]^{[\lambda_{2k}]} e^{-t\hat{F}}]$$

$$-\psi_t t^{|\lambda|+k} \text{tr}^{\text{even}}[a_0[B, a_1]^{(\lambda_1)} \cdots [B, a_{2k}]^{(\lambda_{2k})} e^{-tF}] = O(t^{\frac{3}{2}}) dt. \quad (2.19)$$

This is a local problem and we fix a point x_0 in M_z . Set

$$h(x) = 1 + \frac{1}{2} dt \sum_{j=1}^n x_j c(e_j) \quad (2.20)$$

as in [Zh]. By (5.29) in [Wa3], we have

$$h[F + dt(D + \frac{c(T)}{4})]h^{-1} = F + dtu, \quad (2.21)$$

where the Getzler order $O_G(u) \leq 0$ of u . Write

$$\tilde{A}^{[l]} = [h\hat{F}h^{-1}, \tilde{A}^{[l-1]}], \quad \tilde{A}^{[0]} = A.$$

Then

$$\begin{aligned} & \psi_t t^{|\lambda|+k} \text{tr}^{\text{even}}[a_0[B, a_1]^{[\lambda_1]} \cdots [B, a_{2k}]^{[\lambda_{2k}]} e^{-t\hat{F}}] \\ &= \psi_t t^{|\lambda|+k} \text{tr}^{\text{even}}[a_0[\widetilde{B, a_1}]^{[\lambda_1]} \cdots [\widetilde{B, a_{2k}}]^{[\lambda_{2k}]} e^{-th\hat{F}h^{-1}}]. \end{aligned} \quad (2.22)$$

By the Volterra calculus, we have

$$(\frac{\partial}{\partial t} + F + dtu)^{-1} = (\frac{\partial}{\partial t} + F)^{-1} - dt(\frac{\partial}{\partial t} + F)^{-1}u(\frac{\partial}{\partial t} + F)^{-1}. \quad (2.23)$$

Let

$$a_0[\widetilde{B, a_1}]^{[\lambda_1]} \cdots [\widetilde{B, a_{2k}}]^{[\lambda_{2k}]} = A_0 + dtA_1, \quad (2.24)$$

where

$$A_0 = a_0[B, a_1]^{(\lambda_1)} \cdots [B, a_{2k}]^{(\lambda_{2k})}.$$

Then

$$\begin{aligned} & a_0[\widetilde{B, a_1}]^{[\lambda_1]} \cdots [\widetilde{B, a_{2k}}]^{[\lambda_{2k}]} (\frac{\partial}{\partial t} + F + dtu)^{-1} - A_0(\frac{\partial}{\partial t} + F)^{-1} \\ &= -A_0 dt(\frac{\partial}{\partial t} + F)^{-1}u(\frac{\partial}{\partial t} + F)^{-1} + dtA_1(\frac{\partial}{\partial t} + F)^{-1}. \end{aligned} \quad (2.25)$$

By (2.24) in [Wa3], in order to prove (2.19), we only need to prove

$$t^{k+|\lambda|} \psi_t \text{tr}^{\text{even}}[A_0(\frac{\partial}{\partial t} + F)^{-1}u(\frac{\partial}{\partial t} + F)^{-1}] = O(t^{\frac{3}{2}}), \quad (2.26)$$

$$t^{k+|\lambda|} \psi_t \text{tr}^{\text{even}}[A_1(\frac{\partial}{\partial t} + F)^{-1}] = O(t^{\frac{3}{2}}). \quad (2.27)$$

We note that $A_0(\frac{\partial}{\partial t} + F)^{-1}u(\frac{\partial}{\partial t} + F)^{-1} \in \text{End}^-(\wedge^*(TX) \otimes S(TZ))$, so when we take tr^{even} , only the coefficient of $c(e_1) \cdots c(e_n)$ is left and other terms are zero. Note that

$$O_G(t^{k+|\lambda|} A_0(\frac{\partial}{\partial t} + F)^{-1}u(\frac{\partial}{\partial t} + F)^{-1}) \leq -4, \quad (2.28)$$

so by Lemma 2.5 (1) for $j = n$ odd and $m = -4$ and taking tr^{even} , we get (2.26). By $O_G(u) \leq 0$ and (2.24), we get $O_G(t^{|\lambda|+k} A_1(\frac{\partial}{\partial t} + F)^{-1}) \leq -4$. Again $j = n$, so we get (2.27). Thus we prove Lemma 2.6. \square

Remark. We also introduce a new Bismut superconnection on $\widetilde{M} = M \times \mathbf{R}_+ \rightarrow X \times \mathbf{R}_+$ as in [BGV, Thm. 10.32] and prove a formula which is similar to (2.10). Then we can give a new proof of Lemma 2.6 as in [BGV, p. 347].

For the idempotent $p \in \mathcal{M}_r(C^\infty(M))$, its Chern character $\text{Ch}(p)$ in entire cyclic homology is defined by the formula (for more details see [GS]):

$$\text{Ch}(p) = \text{Tr}(p) + \sum_{k \geq 1} \frac{(-1)^k (2k)!}{k!} \text{Tr}_{2k}((p - \frac{1}{2}) \otimes \bar{p}^{\otimes 2k}) \quad (2.29)$$

where

$$\text{Tr}_{2k} : \mathcal{M}_r(C^\infty(M)) \otimes (\mathcal{M}_r(C^\infty(M))/\mathcal{M}_r(\mathbf{C}))^{\otimes 2k} \rightarrow C^\infty(M) \otimes (C^\infty(M)/\mathbf{C})^{\otimes 2k}$$

is the generalized trace map. Let

$$\|dp\| = \|[B, p]\| = \sum_{i,j} \|d_M p_{i,j}\| \quad (2.30)$$

where $p_{i,j}$ ($1 \leq i, j \leq r$) is the entry of p . Similar to Proposition 2.17 in [Wa1], we have

Proposition 2.7 *Suppose that all D_z are invertible with λ the smallest positive eigenvalue of all $|D_z|$. We assume that $\|dp\| < \lambda$, then the pairing $\langle \hat{\eta}^*(B), \text{Ch}_*(p) \rangle$ is well-defined.*

3 The family index pairing for manifolds with boundary

In this section, we decompose the eta form as a pairing of the eta cochain form with the Chern character of an idempotent matrix and we also decompose the Chern character of the index bundle for a fibration with boundary as a pairing of the family Chern-Connes character for a manifold with boundary with the Chern character of an idempotent matrix.

Suppose that all D_z are invertible with λ the smallest positive eigenvalue of all $|D_z|$ and $\|dp\| < \lambda$. Let $H = \Gamma(M, \wedge^*(TX) \otimes S(TZ))$ and

$$\begin{aligned} p(B \otimes I_r)p : p(H \otimes \mathbf{C}^r) &= L^2(M, \wedge^*(TX) \otimes S(TZ) \otimes p(\mathbf{C}^r)) \\ &\rightarrow L^2(M, \wedge^*(TX) \otimes S(TZ) \otimes p(\mathbf{C}^r)) \end{aligned}$$

be the Bismut superconnection with the coefficient from $F = p(\mathbf{C}^r)$. Then we have

Theorem 3.1 *Under the assumption as above, we have up to an exact form on X*

$$\widehat{\eta}(p(B \otimes I_r)p) = \langle \widehat{\eta}^*(B), \text{Ch}_*(p) \rangle, \quad (3.1)$$

where the left term is the Bismut-Cheeger eta form.

Let

$$\mathbf{B} = \begin{bmatrix} 0 & -B \otimes I_r \\ B \otimes I_r & 0 \end{bmatrix}; \quad \mathbf{p} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}; \quad \sigma = \sqrt{-1} \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix}$$

be operators from $H \otimes \mathbf{C}^r \oplus H \otimes \mathbf{C}^r$ to itself, then

$$\mathbf{B}\sigma = -\sigma\mathbf{B}; \quad \sigma\mathbf{p} = \mathbf{p}\sigma. \quad (3.2)$$

Moreover $\mathbf{B}e^{t\mathbf{B}^2}$ and $e^{t\mathbf{B}^2}$ ($t > 0$) are traceclass. For $u \in [0, 1]$, let

$$B_u = (1-u)B + u[pBp + (1-p)B(1-p)] = B + u(2p-1)[B, p], \quad (3.3)$$

then

$$\mathbf{B}_u = \begin{bmatrix} 0 & -B_u \\ B_u & 0 \end{bmatrix} = \mathbf{B} + u(2p-1)[\mathbf{B}, p]. \quad (3.4)$$

We consider the infinite dimensional bundle $H \otimes \mathbf{C}^r \oplus H \otimes \mathbf{C}^r$ on $X \times [0, 1] \times \mathbf{R} \times [0, \infty)$, parameterized by (b, u, s, t) . Let

$$\widetilde{\mathbf{B}} = t^{\frac{1}{2}}\psi_t\mathbf{B}_u + s\sigma(\mathbf{p} - \frac{1}{2}), \quad (3.5)$$

then $A = d_{(u,s,t)} + \widetilde{\mathbf{B}}$ be a superconnection on $H \otimes \mathbf{C}^r \oplus H \otimes \mathbf{C}^r$. Direct computations show that

$$\begin{aligned} (d + \widetilde{\mathbf{B}})^2 &= t\psi_t\mathbf{B}_u^2 - s^2/4 - (1-u)t^{\frac{1}{2}}s\sigma[\psi_t\mathbf{B}, p] + ds\sigma(p - \frac{1}{2}) \\ &\quad + t^{\frac{1}{2}}du(2p-1)[\psi_t\mathbf{B}, p] + \frac{1}{2}t^{-\frac{1}{2}}dt\psi_t[\mathbf{D}_u + \frac{c(T)}{4}]. \end{aligned} \quad (3.6)$$

We also consider A as A_t , which is a family superconnection parameterized by t on the superbundle with the base $X \times [0, 1] \times \mathbf{R}$ and the fibre $H \otimes \mathbf{C}^r \oplus H \otimes \mathbf{C}^r$. Let $\Gamma_u = \{u\} \times \mathbf{R} \subset [0, 1] \times \mathbf{R}$ be a contour oriented in the direction of increasing s and $\gamma_s = [0, 1] \times \{s\}$ be a contour oriented in the direction of increasing u . By the Duhamel principle and the Stokes theorem as in page 225 in [Wa1], then

$$d_X\omega = \int_{[0,1] \times \mathbf{R}} d \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) = \left(\int_{\Gamma_1} - \int_{\Gamma_0} - \int_{\gamma_{+\infty}} + \int_{\gamma_{-\infty}} \right) \left[\int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) \right], \quad (3.7)$$

where Str^{even} denotes taking the supertrace with value in $\Omega^{\text{even}}(X) \otimes \Omega([0, 1] \times \mathbf{R})$. So in the cohomology of X , we have

$$\int_{\Gamma_0} \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) = \int_{\Gamma_1} \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}). \quad (3.8)$$

Similar to (3.8) in [Wa1], we have

$$\int_{\Gamma_0} \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) = -4\sqrt{-1}\pi[\langle \hat{\eta}^*(B), \text{Ch}(p) \rangle - \langle \hat{\eta}^*(B), \text{rk}(p)\text{Ch}_*(1) \rangle]. \quad (3.9)$$

Similar to (3.10) in [Wa1], we have

$$\begin{aligned} \int_{\Gamma_1} \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) &= -2\sqrt{-1} \int_{-\infty}^{+\infty} e^{-s^2/4} ds \\ &\cdot \int_0^{+\infty} \psi_t \text{Tr}^{\text{even}}[(p - \frac{1}{2})(D_1 + \frac{c(T)}{4})e^{-tB_1^2}] d\sqrt{t}. \end{aligned} \quad (3.10)$$

By the following lemma 3.2 and (3.8)-(3.10), similar to (3.12) in [Wa1], we can prove Theorem 3.1. \square

Lemma 3.2 *Let $B_s = B + s(2p - 1)[B, p]$ for $s \in [0, 1]$. We assume that all D_z be invertible and $\|d_{Mp}\| < \lambda$, then we have $\hat{\eta}(B_0) = \hat{\eta}(B_1)$.*

Proof. By $\|d_{Mp}\| < \lambda$, then $D_s = D + s(2p - 1)[D, p]$ is invertible for $s \in [0, 1]$. Similar to the discussions of Proposition 4.4 in [Wu], the eta form of B_s is well defined. So $\hat{\eta}(B_s)$ is smooth. Let $B_s = D_s + A_{[1]} - \frac{c(T)}{4}$ and $A_0 = (2p - 1)[D, p]$. Then by the definition of the eta form and the Duhamel principle, we have

$$\frac{d}{ds} \hat{\eta}(B_s) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \psi_t \text{tr}^{\text{even}}[A_0 e^{-tB_s^2}] d\sqrt{t} + L, \quad (3.11)$$

where

$$\begin{aligned} L &= -\frac{1}{\sqrt{\pi}} \int_0^{+\infty} \psi_t \text{tr}^{\text{even}} \left\{ t(B_s + \frac{c(T)}{2} - A_{[1]}) \right. \\ &\cdot \left. \int_0^1 e^{-\sigma t B_s^2} [(2p - 1)[B, p], B_s]_+ e^{-(1-\sigma)t B_s^2} d\sigma \right\} d\sqrt{t}. \end{aligned} \quad (3.12)$$

By $\text{tr}^{\text{even}}(AB) = \text{tr}^{\text{even}}(BA)$ and $B_s e^{-\sigma B_s^2} = e^{-\sigma B_s^2} B_s$, we have

$$\begin{aligned} &\text{tr}^{\text{even}} \left\{ B_s \int_0^1 e^{-\sigma t B_s^2} [(2p - 1)[B, p], B_s]_+ e^{-(1-\sigma)t B_s^2} d\sigma \right\} \\ &= \int_0^1 \text{tr}^{\text{even}} \left\{ (2p - 1)[B, p] e^{-\sigma t B_s^2} [B_s, B_s]_+ e^{-(1-\sigma)t B_s^2} d\sigma \right\}, \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} &\text{tr}^{\text{even}} \left\{ \left(\frac{c(T)}{2} - A_{[1]} \right) \int_0^1 e^{-\sigma t B_s^2} [(2p - 1)[B, p], B_s]_+ e^{-(1-\sigma)t B_s^2} d\sigma \right\} \\ &= \int_0^1 \text{tr}^{\text{even}} \left\{ (2p - 1)[B, p] e^{-\sigma t B_s^2} \left[\frac{c(T)}{2} - A_{[1]}, B_s \right]_+ e^{-(1-\sigma)t B_s^2} d\sigma \right\}. \end{aligned} \quad (3.14)$$

By (3.12)-(3.14)

$$L = - \int_0^{+\infty} \frac{\sqrt{t}}{2\sqrt{\pi}} \psi_t \int_0^1 \text{tr}^{\text{even}} \left\{ (2p - 1)[B, p] e^{-\sigma t B_s^2} \left[D_s + \frac{c(T)}{4}, B_s \right]_+ e^{-(1-\sigma)t B_s^2} d\sigma \right\} dt, \quad (3.15)$$

By

$$\frac{d(t\psi_t B_s^2)}{dt} = \frac{1}{2}\psi_t[D_s + \frac{c(T)}{4}, B_s]_+, \quad (3.16)$$

(3.11) and (3.15), using the Duhamel principle and the Leibniz rule, then we get

$$\frac{\partial}{\partial t} \left\{ \frac{\sqrt{t}}{\sqrt{\pi}} \psi_t \text{tr}^{\text{even}} \left[(2p-1)[D, p] e^{-tB_s^2} \right] \right\} = \frac{\partial}{\partial s} \left\{ \frac{1}{2\sqrt{\pi t}} \psi_t \text{tr}^{\text{even}} \left[\left(D_s + \frac{c(T)}{4} \right) e^{-tB_s^2} \right] \right\}. \quad (3.17)$$

So

$$\frac{d}{ds} \hat{\eta}(B_s) = \frac{\sqrt{t}}{\sqrt{\pi}} \psi_t \text{tr}^{\text{even}} \left[(2p-1)[D, p] e^{-tB_s^2} \right] \Big|_0^{+\infty}. \quad (3.18)$$

By D_s being invertible, $\text{tr}^{\text{even}} \left[(2p-1)[D, p] e^{-tB_s^2} \right]$ exponentially decays, so

$$\lim_{t \rightarrow +\infty} \frac{\sqrt{t}}{\sqrt{\pi}} \psi_t \text{tr}^{\text{even}} \left[(2p-1)[D, p] e^{-tB_s^2} \right] = 0. \quad (3.19)$$

By Lemma 2.3, similar to the discussions on page 164 in [Wu], we have

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{\sqrt{t}}{\sqrt{\pi}} \psi_t \text{tr}^{\text{even}} \left[(2p-1)[D, p] e^{-tB_s^2} \right] \\ &= c_0 \int_Z \hat{A}(TZ) \text{tr} \left\{ (2p-1)(d_Z p) \exp \left[\frac{\sqrt{-1}}{2\pi} (A' \wedge A' + dA') \right] \right\} = 0, \end{aligned} \quad (3.20)$$

where $A' = s(2p-1)d_M p$. Then by (3.18)-(3.20), we prove Lemma 3.2. \square

Let N be a fibration with the even-dimensional compact spin fibre. Let M be the boundary of N . We endow N with a metric which is a product in a collar neighborhood of M . Denote by B (B_M) the Bismut superconnection on N (M). Let $C_*^\infty(N) = \{f \in C^\infty(N) | f \text{ is independent of the normal coordinate } x_n \text{ near the boundary}\}$.

Definition 3.3 The family Chern-Connes character on N , $\tau = \{\tau_0, \tau_2, \dots, \tau_{2q} \dots\}$ is defined by

$$\begin{aligned} \tau_{2q}(B)(f^0, f^1, \cdot, f^{2q}) &:= -\hat{\eta}_{2q}(B_M)(f^0|_M, f^1|_M, \cdot, f^{2q}|_M) \\ &+ \frac{1}{(2q)!(2\pi\sqrt{-1})^q} \int_Z \hat{A}(TZ) f^0 df^1 \wedge \dots \wedge df^{2q}, \end{aligned} \quad (3.21)$$

where $f^0, f^1, \cdot, f^{2q} \in C_*^\infty(N)$.

Similar to Proposition 4.2 in [Wal], we have

Proposition 3.4 *The family Chern-Connes character is $b - \tilde{B}$ closed (Here we use \tilde{B} instead of the Connes operator B . For the definitions of b , \tilde{B} , see [Co]). That is, in the cohomology of X , we have*

$$b\tau_{2q-2} + \tilde{B}\tau_{2q} = 0. \quad (3.22)$$

By Proposition 2.7, we have

Proposition 3.5 *Suppose that all $D_{M,z}$ are invertible with λ the smallest positive eigenvalue of all $|D_{M,z}|$. We assume that $\|d(p|_M)\| < \lambda$, then the pairing $\langle \tau, \text{Ch}(p) \rangle$ is well-defined.*

We let $C_1(M) = M \times (0, 1]$, $\tilde{N} = N \cup_{M \times \{1\}} C_1(M)$ and \mathcal{U} be a collar neighborhood of M in N . For $\varepsilon > 0$, we take a metric g^ε of \tilde{N} such that on $\mathcal{U} \cup_{M \times \{1\}} C_1(M)$

$$g^\varepsilon = \frac{dr^2}{\varepsilon} + r^2 g^M.$$

Let $S = S^+ \oplus S^-$ be spinors bundle associated to $(\tilde{N}_z, g^\varepsilon)$ and H^∞ be the set $\{\xi \in \Gamma(\tilde{N}_z, S) \mid \xi \text{ and its derivatives are zero near the vertex of cone}\}$. Denote by $L_c^2(\tilde{N}_z, S)$ the L^2 -completion of H^∞ (similar define $L_c^2(\tilde{N}_z, S^+)$ and $L_c^2(\tilde{N}_z, S^-)$). Let

$$D_{z,\varepsilon} : H^\infty \rightarrow H^\infty; \quad D_{z,+, \varepsilon} : H_+^\infty \rightarrow H_-^\infty,$$

be the Dirac operators associated to $(\tilde{N}_z, g^\varepsilon)$ which are Fredholm operators for the sufficient small ε . When $D_{M,z}$ is invertible, the index bundle of $\{D_z\}$ is well defined by [BC]. We recall the Bismut-Cheeger family index theorem for the twisting bundle Imp with the connection pd in [BC]

$$\text{ch}[\text{Ind}(pD_{z,+, \varepsilon} p)] = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(2\pi\sqrt{-1})^r} \int_Z \hat{A}(TZ) \text{Tr}[p(dp)^{2r}] - \hat{\eta}(pB_M p). \quad (3.22)$$

Then we get

Theorem 3.6 *Suppose that all $D_{M,z}$ are invertible with λ the smallest positive eigenvalue of all $|D_{M,z}|$. We assume that $\|d(p|_M)\| < \lambda$ and $p \in M_{r \times r}(C_*^\infty(N))$, then in the cohomology of X*

$$\text{ch}[\text{Ind}(pD_{z,+, \varepsilon} p)] = \langle \tau(B), \text{Ch}(p) \rangle. \quad (3.23)$$

Proof. Let $\hat{\tau}_{2q}$ be defined by

$$\hat{\tau}_{2q}(B)(f^0, f^1, \cdot, f^{2q}) := \frac{1}{(2q)!(2\pi\sqrt{-1})^q} \int_Z \hat{A}(TZ) f^0 df^1 \wedge \cdots \wedge df^{2q}, \quad (3.24)$$

where $f^0, f^1, \cdot, f^{2q} \in C_*^\infty(N)$. Recall

$$\text{ch}(\text{Imp}) = \sum_{q=0}^{\infty} \frac{(-1)^q}{(2\pi\sqrt{-1}q!)} \text{Tr}[p(dp)^{2q}], \quad (3.25)$$

By (3.22) and Theorem 3.1 and (2.29), (3.24) and $\text{Tr}[(dp)^{2k}] = 0$ for $1 \leq k$, we have

$$\langle \hat{\tau}_*(B), \text{ch}(p) \rangle = \int_Z \hat{A}(TZ) \text{ch}(\text{Imp}), \quad (3.26)$$

so Theorem 3.6 holds.

□

4 The family b -Chern-Connes character

In this section, we define the family b -Chern-Connes character which is the family version of the Getzler's b -Chern-Connes character in [Ge2] and then we prove that it is entire and give its variation formula.

Let us recall the exact b -geometry (see [LMP],[Xi]). Let N be a compact fibration with boundary M and denote by N° its interior of N . We take the b -metric $g_b = \frac{1}{r^2}dr \otimes dr + g_M$ near the M where r is the normal coordinate near the boundary. Let $x = \ln r$ which gives an isometry between the infinite cylinder $((-\infty, c_0] \times M, g_{cyl} = dx \otimes dx + g_M)$ and the collar neighborhood U with the exact b -metric. Now we consider the complete Riemannian manifold $\hat{N} = (-\infty, c_0] \times M \cup_M (N \setminus U^\circ)$ instead of N° with the exact b -metric. Let $C_{\text{exp}}^\infty(\hat{N})$ be the space of smooth functions on \hat{N} which expands exponentially on the infinite cylinder $(-\infty, c_0] \times M$. A smooth function $f \in C^\infty(\hat{N})$ expands exponentially on $(-\infty, c_0] \times M$ if $f(x, y) \sim \sum_{k=0}^{\infty} e^{kx} f_k(y)$ for $(x, y) \in (-\infty, c_0] \times M$, where $f_k(y) \in C^\infty(M)$ for each k . That is

$$f(x, y) - \sum_{k=0}^{N-1} e^{kx} f_k(y) = e^{Nx} R_N(x, y), \quad (4.1)$$

where all derivative of $R_N(x, y)$ in x and y are bounded.

On $(-\infty, c_0] \times M$, we write $a = a_c + e^x a_\infty$ for $a \in C_{\text{exp}}^\infty(\hat{N})$ with $a_c, a_\infty \in C^\infty(\hat{N})$ and a_c constant with respect to x . Following [Xi], define the b -norm of a by $^b||a|| := ||a_c||_1 + 2||a_\infty||_1$. The b -integral of a along the fibre is defined by

$$\int^b a d\text{vol} := \int_{N_z \setminus U_z^\circ} a|_{N_z \setminus U_z^\circ} d\text{vol} + \int_{(-\infty, c_0] \times M_z} e^x a_\infty d\text{vol}. \quad (4.2)$$

Following [LMP, A.1] and [MP1], we can define the b -pseudodifferential operator with coefficients in $\wedge^*(TX)$ and the pointwise trace of the Schwartz kernel of smooth b -pseudodifferential operators is a b -function. We define the b -trace is the b -integral of this b -function. That is, for $A \in \Psi_b^{-\infty}(\hat{N}_z, \wedge^*(TX) \otimes S(T\hat{N}_z))$ and its Schwartz kernel k_A , define the b -trace which is in $\Omega(X)$ by

$$^b\text{Str}(A) = \int^b \text{Str}(k_A(x, x)) d\text{vol}. \quad (4.3)$$

Let B be the Bismut superconnection on \widehat{N} and $B_t = t\psi_t B$ and $F_t = B_t^2$. By [MP1], $e^{-F_t} \in \Psi_b^{-\infty}(\widehat{N}_z, \wedge^*(TX) \otimes S(T\widehat{N}_z))$. For $A_0, \dots, A_q \in \Psi_b^\infty(\widehat{N}, \wedge^*(TX) \times S(T\widehat{N}_z))$, we define

$$\langle\langle A_0, \dots, A_q \rangle\rangle_b = \int_{\Delta_q} {}^b\text{Str}[A_0 e^{-\sigma_0 F} A_1 e^{-\sigma_1 F} \dots A_q e^{-\sigma_q F}] d\sigma, \quad (4.4)$$

and

$$\langle\langle A_0, \dots, A_q \rangle\rangle_{b,t} = \int_{\Delta_q} {}^b\text{Str}[A_0 e^{-\sigma_0 F_t} A_1 e^{-\sigma_1 F_t} \dots A_q e^{-\sigma_q F_t}] d\sigma. \quad (4.5)$$

For $f_0, \dots, f_k \in C_{\text{exp}}^\infty(\widehat{N})$, we define the **family b -Chern-Connes character** by

$${}^b\text{ch}^k(B)(f_0, \dots, f_k) := \langle\langle f_0, [B, f_1], \dots, [B, f_k] \rangle\rangle_b; \quad (4.6)$$

$${}^b\text{ch}^k(B_t)(f_0, \dots, f_k) := \langle\langle f_0, [B_t, f_1], \dots, [B_t, f_k] \rangle\rangle_{b,t}. \quad (4.7)$$

Define

$${}^b\text{ch}^k(B, V) := \sum_{0 \leq j \leq k} (-1)^{j \deg V} \langle\langle f_0, [B, f_1], \dots, [B, f_j], V, [B, f_{j+1}], \dots, [B, f_k] \rangle\rangle_b. \quad (4.8)$$

Similarly we may define ${}^b\text{ch}^k(B_t, V)$. The family b -Chern-Connes character is well defined by the following Proposition 4.7. We recall the following lemma

Lemma 4.1 ([MP1, Proposition 9]) *For $A \in \Psi_{b,cl}^\infty(\widehat{N}, \wedge^*(TX) \times S(T\widehat{N}_z))$ and $L \in \Psi_{b,cl}^{-\infty}(\widehat{N}, \wedge^*(TX) \times S(T\widehat{N}_z))$, we have*

$${}^b\text{Str}[A, L] = \frac{\sqrt{-1}}{2\pi} \int_{-\infty}^{+\infty} \text{Str}_M \left(\frac{\partial I(A, \lambda)}{\partial \lambda} \cdot I(L, \lambda) \right) d\lambda, \quad (4.9)$$

where $I(L, \lambda)$ is the indicial family of L (for the definition, see [LMP] or [MP1]).

Let \mathbf{D} be the Dirac operator on the cylinder $(-\infty, +\infty) \times M$, then $\mathbf{D} = c(dx) \frac{d}{dx} + \mathbf{D}_\partial$. On the boundary, $c(dx)$ gives a natural identification of the even and odd half spinor bundle, then with respect to this splitting

$$\mathbf{D} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & D_\partial \\ D_\partial & 0 \end{pmatrix}. \quad (4.10)$$

By [MP1, p.139], we have

Lemma 4.2 *The following equality holds*

$$I(B, \lambda) = \sqrt{-1} c(dx) \lambda + B'^M; \quad I(F, \lambda) = \lambda^2 + (B'^M)^2 \quad (4.11)$$

where with respect the above splitting

$$B'^M = D_\partial \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_{\alpha=1}^q f_\alpha^* \wedge (\nabla_{f_\alpha}^{S(TM/X)} + \frac{1}{2} k^M(f_\alpha)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$-\frac{1}{4}c(T^M) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.12)$$

By Lemma 4.2, we have

$$F'^M := (B'^M)^2 \in \Omega^{\text{even}}(X) \begin{pmatrix} L_1 & 0 \\ 0 & L_1 \end{pmatrix} + \Omega^{\text{odd}}(X) \begin{pmatrix} 0 & L_2 \\ L_2 & 0 \end{pmatrix}, \quad (4.13)$$

where $L_1, L_2 \in \text{End}(S(TM_z))$. Similarly, we have

$$I([B, a], \lambda) = \begin{pmatrix} 0 & [D_{\partial}, a_{\partial}] \\ [D_{\partial}, a_{\partial}] & 0 \end{pmatrix} + d_X a_{\partial} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.14)$$

By Lemma 4.1 and Lemma 4.2, we have

Lemma 4.3 *For $K \in \Psi_{b,cl}^{-\infty}(\hat{N}, \wedge^*(TX) \times S(T\hat{N}_z))$, we have*

$${}^b\text{Str}[B, K] = d_X {}^b\text{Str}(K) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Str}_M[c(dx)I(K, \lambda)]d\lambda. \quad (4.15)$$

By Lemmas 4.1-4.3, similar to Lemma 6.3 in [Ge2], we have

Lemma 4.4 *Let $A_j \in \Psi_{b,cl}^{\infty}(\hat{N}, \wedge^*(TX) \times S(T\hat{N}_z))$ which indicial family is independent of λ .*

1. *If $\varepsilon_j = (|A_0| + \dots + |A_{j-1}|)(|A_j| + \dots + |A_k|)$, then*

$$\langle\langle A_0, \dots, A_k \rangle\rangle_{b,t} = (-1)^{\varepsilon_j} \langle\langle A_j, \dots, A_k, A_0, \dots, A_{j-1} \rangle\rangle_{b,t}. \quad (4.16)$$

2.

$$\langle\langle A_0, \dots, A_k \rangle\rangle_{b,t} = \sum_{j=0}^k (-1)^{\varepsilon_j} \langle\langle 1, A_j, \dots, A_k, A_0, \dots, A_{j-1} \rangle\rangle_{b,t}. \quad (4.17)$$

3.

$$\begin{aligned} -d_X \langle\langle A_0, \dots, A_k \rangle\rangle_{b,t} + \sum_{j=0}^k (-1)^{|A_0| + \dots + |A_{j-1}|} \langle\langle A_0, \dots, [B_t, A_j], \dots, A_k \rangle\rangle_{b,t} \\ = \langle\langle (A_0)_{\partial}, \dots, (A_k)_{\partial} \rangle\rangle_{\partial,t}, \end{aligned} \quad (4.18)$$

where when $\dim M_z$ is odd,

$$\langle\langle (A_0)_{\partial}, \dots, (A_k)_{\partial} \rangle\rangle_{\partial,t} := \frac{-1}{2\sqrt{\pi}} \int_{\Delta_k} \text{Str}_M \left[c(dx) A_{0,\partial} e^{-\sigma_0 F_t'^M} \dots A_{k,\partial} e^{-\sigma_k F_t'^M} \right] d\sigma. \quad (4.19)$$

4. *For $0 \leq j < k$,*

$$\langle\langle A_0, \dots, [F_t, A_j], \dots, A_k \rangle\rangle_{b,t}$$

$$= -\langle\langle A_0, \dots, A_j A_{j+1}, \dots, A_k \rangle\rangle_{b,t} + \langle\langle A_0, \dots, A_{j-1} A_j, \dots, A_k \rangle\rangle_{b,t}. \quad (4.20)$$

For $j = k$,

$$\begin{aligned} & \langle\langle A_0, \dots, A_{k-1}, [F_t, A_k] \rangle\rangle_{b,t} \\ &= \langle\langle A_0, \dots, A_{k-2}, A_{k-1} A_k \rangle\rangle_{b,t} - (-1)^{\varepsilon_k} \langle\langle A_k A_0, A_1, \dots, A_{k-1} \rangle\rangle_{b,t}. \end{aligned} \quad (4.21)$$

Proof. 1) By the definition of trace, we have

$$\begin{aligned} & \langle\langle A_0, \dots, A_k \rangle\rangle_{b,t} - (-1)^{\varepsilon_j} \langle\langle A_j, \dots, A_k, A_0, \dots, A_{j-1} \rangle\rangle_{b,t} \\ &= \int_{\Delta_k} {}^b\text{str} \left[A_0 e^{-\sigma_0 F_t} A_1 \dots A_{j-1} e^{-\sigma_{j-1} F_t}, A_j e^{-\sigma_j F_t} A_{j+1} \dots A_k e^{-\sigma_k F_t} \right] d\sigma. \end{aligned} \quad (4.22)$$

By $I(F_t, \lambda) = t\lambda^2 + F_t^\partial$ and Lemma 4.1 and $\int_{-\infty}^{+\infty} \lambda e^{-t\lambda^2} d\lambda = 0$, we know that 1) holds.

(2) (2) comes from (1) by the same trick in the [Get2, p.18].

(3) By $B_t e^{F_t} = e^{F_t} B_t$, we have

$$\begin{aligned} & \sum_{j=0}^k (-1)^{|A_0|+\dots+|A_{j-1}|} \langle\langle A_0, \dots, [B_t, A_j], \dots, A_k \rangle\rangle_{b,t} \\ &= {}^b\text{Str}[B_t, A_0 e^{-s_1 F_t} A_1 e^{-(s_2-s_1)F_t} \dots A_k e^{-(1-s_k)F_t}]. \end{aligned} \quad (4.23)$$

Then by Lemma 4.3 and (4.23), we get (3).

(4) By the Duhamel principle, we have

$$\frac{d}{ds_j} [e^{-(s_j-s_{j-1})F_t} A_j e^{-(s_{j+1}-s_j)F_t}] = -e^{-(s_j-s_{j-1})F_t} [F_t, A_j] e^{-(s_{j+1}-s_j)F_t}. \quad (4.24)$$

By the integration along $\int_{s_{j-1}}^{s_{j+1}}$, we have

$$[e^{-(s_{j+1}-s_{j-1})F_t}, A_j] = - \int_{s_{j-1}}^{s_{j+1}} e^{-(s_j-s_{j-1})F_t} [F_t, A_j] e^{-(s_{j+1}-s_j)F_t} ds_j. \quad (4.25)$$

By (4.25), we get (4.20). By (4.25) and the indicial family of A_k is independent of λ and Lemma 4.1, we get (4.21). \square

By Lemma 4.4, similar to the proof of Theorem 6.2 in [Ge2], we get

Theorem 4.5 *When $\dim M_z$ is odd, for any $k \geq 0$, the following equality holds*

$$b^{\text{ch}}{}^{k-2}(B_t) + \widetilde{B}^{\text{ch}}{}^k(B_t) - d_X b^{\text{ch}}{}^{k-1}(B_t) = \widetilde{Ch}^{k-1}(B_t'^M) \circ i_M^*, \quad (4.26)$$

where

$$\widetilde{Ch}^{k-1}(B_t'^M) \circ i_M^*(a_0, \dots, a_{k-1}) = \langle\langle (a_0)_\partial, [B_t'^M, a_{1,\partial}], \dots, [B_t'^M, a_{k-1,\partial}] \rangle\rangle_{\partial,t}. \quad (4.27)$$

Proof. By Lemma 4.4 (3), for $A_0 = a_0$, $A_j = [B_t, a_j]$ and $1 \leq j \leq k-1$ we have

$$\begin{aligned} & -d_X \langle \langle a_0, [B_t, a_1], \dots, [B_t, a_{k-1}] \rangle \rangle_{b,t} + \langle \langle [B_t, a_0], [B_t, a_1], \dots, [B_t, a_{k-1}] \rangle \rangle_{b,t} \\ & + \sum_{j=1}^{k-1} (-1)^{j-1} \langle \langle a_0, [B_t, a_1], \dots, [B_t^2, a_j], \dots, [B_t, a_{k-1}] \rangle \rangle_{b,t} \\ & = \langle \langle (a_0)_\partial, [B_t'^M, a_{1,\partial}], \dots, [B_t'^M, a_{k-1,\partial}] \rangle \rangle_{\partial,t}. \end{aligned} \quad (4.28)$$

By the definition of \tilde{B} (see [Co]) and Lemma 4.4 (2), we get

$$\tilde{B}^{\text{bch}}{}^k(B_t)(a_0, \dots, a_{k-1}) = \langle \langle [B_t, a_0], [B_t, a_1], \dots, [B_t, a_{k-1}] \rangle \rangle_{b,t}. \quad (4.29)$$

By Lemma 4.4 (4), and $[B_t, a_j a_{j+1}] = [B_t, a_j] a_{j+1} + a_j [B_t, a_{j+1}]$, we have

$$b^{\text{bch}}{}^{k-2}(B_t)(a_0, \dots, a_{k-1}) = \sum_{j=1}^{k-1} (-1)^{j-1} \langle \langle a_0, [B_t, a_1], \dots, [B_t^2, a_j], \dots, [B_t, a_{k-1}] \rangle \rangle_{b,t}. \quad (4.30)$$

By (4.28)-(4.30), we get Theorem 4.5. \square

By Theorem 4.5, we have

Theorem 4.6 *When $\dim N_z$ is even and $k-1$ is even, the following equality holds*

$$\begin{aligned} & \frac{d^{\text{bch}}{}^{k-1}(B_t)}{dt} + b^{\text{bch}}{}^{k-2}(B_t, \frac{dB_t}{dt}) + \tilde{B}^{\text{bch}}{}^k(B_t, \frac{dB_t}{dt}) \\ & + d_X b^{\text{bch}}{}^{k-1}(B_t, \frac{dB_t}{dt}) = -\frac{1}{\sqrt{\pi}} \text{ch}^{k-1}(B_t^M, \frac{dB_t^M}{dt}). \end{aligned} \quad (4.31)$$

Proof. We know that B_t is a superconnection on the infinite dimensional bundle $C^\infty(N, E) \rightarrow X$ which we write $\mathcal{E} \rightarrow X$. Let $\tilde{X} = X \times \mathbf{R}_+$, and $\tilde{\mathcal{E}}$ be the superbundle $\pi^* \mathcal{E}$ over \tilde{X} , which is the pull-back to \tilde{X} of \mathcal{E} . Define a superconnection \hat{B} on $\tilde{\mathcal{E}}$ by the formula

$$(\hat{B}\beta)(y, t) = (B_t \beta(\cdot, t))(y) + dt \wedge \frac{\partial \beta(y, t)}{\partial t}. \quad (4.32)$$

The curvature $\hat{\mathcal{F}}$ of \hat{B} is

$$\hat{\mathcal{F}} = \mathcal{F}_t - \frac{dB_t}{dt} \wedge dt, \quad (4.33)$$

where $\mathcal{F}_t = B_t^2$ is the curvature of B_t . By the Duhamel principle, then

$$e^{-\hat{\mathcal{F}}} = e^{-\mathcal{F}_t} - dt \left(\int_0^1 e^{-u\mathcal{F}_t} \frac{dB_t}{dt} e^{-(1-u)\mathcal{F}_t} du \right). \quad (4.34)$$

Then for any $l \geq 0$, we have

$${}^b\text{ch}^l(\widehat{B}) = {}^b\text{ch}^l(B_t) - dt {}^b\text{ch}^l(B_t, \frac{dB_t}{dt}). \quad (4.35)$$

By Theorem 4.5, we have

$$b {}^b\text{ch}^{k-2}(\widehat{B}) + \widetilde{B} {}^b\text{ch}^k(\widehat{B}) - d_X {}^b\text{ch}^{k-1}(\widehat{B}) = \widetilde{C} \widetilde{h}^{k-1}(\widehat{B}'^M). \quad (4.36)$$

By Theorem 4.5 and (4.36), (4.35) and $d_{\widetilde{X}} = d_X + dt \frac{d}{dt}$, we have

$$\begin{aligned} dt \left[\frac{d {}^b\text{ch}^{k-1}(B_t)}{dt} + b {}^b\text{ch}^{k-2}(B_t, \frac{dB_t}{dt}) + \widetilde{B} {}^b\text{ch}^k(B_t, \frac{dB_t}{dt}) \right. \\ \left. + d_X {}^b\text{ch}^{k-1}(B_t, \frac{dB_t}{dt}) \right] = \widetilde{C} \widetilde{h}^{k-1}(B_t'^M) - \widetilde{C} \widetilde{h}^{k-1}(\widehat{B}'^M). \end{aligned} \quad (4.37)$$

By (4.19), (4.35), (4.12) and (4.13), we get

$$\widetilde{C} \widetilde{h}^{k-1}(B_t'^M) - \widetilde{C} \widetilde{h}^{k-1}(\widehat{B}'^M) = -\frac{1}{\sqrt{\pi}} dt {}^b\text{ch}^{k-1}(B_t^M, \frac{dB_t^M}{dt}). \quad (4.38)$$

By (4.37) and (4.38), we get (4.31). \square

We recall that an even cochain $\{\Phi_{2n}\}$ is called entire if $\sum_n \|\Phi_{2n}\| n! z^n$ is entire, where $\|\Phi\| := \sup_{b \|f^j\| \leq 1} \|\Phi(f^0, f^1, \dots, f^{2k})\|$ for $f_j \in C_{\text{exp}}^\infty(\widehat{N})$. Then we have

Proposition 4.7 ${}^b\text{ch}(B)$ is an entire cochain and $\langle {}^b\text{ch}(B), \text{ch}(p) \rangle$ is well defined.

Proof. For $A \in \Psi_b^{-\infty}(\widehat{N}_z, \wedge^*(TX) \otimes S(T\widehat{N}_z))$ and its Schwartz kernel k_A , we define

$$\text{Str}^{N \setminus U}(A) = \int_{N_z \setminus U_z} \text{Str}(k_A(x, x)) d\text{vol}; \quad {}^b\text{Str}^{\text{end}}(A) = \int_{(-\infty, c_0) \times M_z}^b \text{Str}(k_A(x, x)) d\text{vol}. \quad (4.39)$$

So for $a_0, \dots, a_q \in C_{\text{exp}}^\infty(\widehat{N})$,

$$\begin{aligned} & \int_{\Delta_q} {}^b\text{Str} \left[a_0 e^{-\sigma_0 F} [B, a_1] e^{-\sigma_1 F} \dots [B, a_q] e^{-\sigma_q F} \right] d\sigma \\ &= \int_{\Delta_q} \text{Str}^{N \setminus U} \left[a_0 e^{-\sigma_0 F} [B, a_1] e^{-\sigma_1 F} \dots [B, a_q] e^{-\sigma_q F} \right] d\sigma \\ &+ \int_{\Delta_q} {}^b\text{Str}^{\text{end}} \left[a_0 e^{-\sigma_0 F} [B, a_1] e^{-\sigma_1 F} \dots [B, a_q] e^{-\sigma_q F} \right] d\sigma. \end{aligned} \quad (4.40)$$

By the discussions on the compact fibration as in [BeC], we have

$$\left| \int_{\Delta_q} \text{Str}^{N \setminus U} \left[a_0 e^{-\sigma_0 F} [B, a_1] e^{-\sigma_1 F} \dots [B, a_q] e^{-\sigma_q F} \right] d\sigma \right| \leq \text{Tr}(e^{-\frac{D^2}{2}})^b \|a_0\|^b \|a_1\| \dots^b \|a_q\|. \quad (4.41)$$

On the cylinder, we get

$$[B, a_j] = C_j + e^x B_j; \quad a_0 = C_0 + e^x B_0, \quad (4.42)$$

where

$$C_j = c(d_{M_z}(a_j)_c) + d_X(a_j)_c; \quad B_j = c(d_{N_z}(a_j)_\infty) + c((a_j)_\infty dx) + d_X(a_j)_\infty,$$

and C_j is constant along the normal direction x . The second term in (4.40) can be written as a sum of terms of the following two types:

- I) $\int_{\Delta_q} {}^b\text{Str}^{\text{end}} \left[C_0 e^{-\sigma_0 F} C_1 e^{-\sigma_1 F} \dots C_q e^{-\sigma_q F} \right] d\sigma$,
 II) $\int_{\Delta_q} {}^b\text{Str}^{\text{end}} \left[A_0 e^{-\sigma_0 F} \dots e^{-\sigma_j F} e^x B_j e^{-\sigma_{j+1} F} \dots A_q e^{-\sigma_q F} \right] d\sigma$, where $A_j = C_j$ or $e^x B_j$.

Firstly we estimate the type I) integral. Without generality, we set $q = 1$. Let $B^2 = D^2 + A_{[+]}$ and D_R be the Dirac operator on the cylinder $(-\infty, c_0) \times M_z$. By the Duhamel principle, we have

$$\begin{aligned} C_0 e^{-\sigma_0 F} C_1 e^{-\sigma_1 F} &= C_0 \sum_{m \geq 0} (-\sigma_0)^m \int_{\Delta_m} e^{-\sigma_0 v_0 D^2} A_{[+]} \dots A_{[+]} e^{-\sigma_0 v_m D^2} dv \\ &\quad \times C_1 \sum_{l \geq 0} (-\sigma_1)^l \int_{\Delta_l} e^{-\sigma_1 v'_0 D^2} A_{[+]} \dots A_{[+]} e^{-\sigma_1 v'_l D^2} dv' \\ &= C_0 \sum_{m \geq 0} (-\sigma_0)^m \int_{\Delta_m} [e^{-\sigma_0 v_0 D^2} - e^{-\sigma_0 v_0 D_R^2}] A_{[+]} \dots A_{[+]} e^{-\sigma_0 v_m D^2} dv \\ &\quad \times C_1 \sum_{l \geq 0} (-\sigma_1)^l \int_{\Delta_l} e^{-\sigma_1 v'_0 D^2} A_{[+]} \dots A_{[+]} e^{-\sigma_1 v'_l D^2} dv' \\ &\quad + \dots + C_0 \sum_{m \geq 0} (-\sigma_0)^m \int_{\Delta_m} e^{-\sigma_0 v_0 D_R^2} A_{[+]} \dots A_{[+]} e^{-\sigma_0 v_m D_R^2} dv \\ &\quad \times C_1 \sum_{l \geq 0} (-\sigma_1)^l \int_{\Delta_l} e^{-\sigma_1 v'_0 D_R^2} A_{[+]} \dots A_{[+]} [e^{-\sigma_1 v'_l D^2} - e^{-\sigma_1 v'_l D_R^2}] dv' \\ &\quad + C_0 \sum_{m \geq 0} (-\sigma_0)^m \int_{\Delta_m} e^{-\sigma_0 v_0 D_R^2} A_{[+]} \dots A_{[+]} e^{-\sigma_0 v_m D_R^2} dv \\ &\quad \times C_1 \sum_{l \geq 0} (-\sigma_1)^l \int_{\Delta_l} e^{-\sigma_1 v'_0 D_R^2} A_{[+]} \dots A_{[+]} e^{-\sigma_1 v'_l D_R^2} dv'. \end{aligned} \quad (4.43)$$

We know that $A_{[+]}$ is independent of x on the cylinder and $D_R^2 = \Delta_R + D_{M_z}^2$, so

$$\begin{aligned} &{}^b\text{Str}^{\text{end}} \left[C_0 \sum_{m \geq 0} (-\sigma_0)^m \int_{\Delta_m} e^{-\sigma_0 v_0 D_R^2} A_{[+]} \dots A_{[+]} e^{-\sigma_0 v_m D_R^2} dv \right. \\ &\quad \left. \times C_1 \sum_{l \geq 0} (-\sigma_1)^l \int_{\Delta_l} e^{-\sigma_1 v'_0 D_R^2} A_{[+]} \dots A_{[+]} e^{-\sigma_1 v'_l D_R^2} dv' \right] \Big|_{(-\infty, c_0) \times M_z} = 0. \end{aligned} \quad (4.44)$$

We estimate the first term K_1 in (4.43) and the estimate of other terms is similar. Since D and D_R are self adjoint, we can apply the functional calculus to these two operators. Then $\|e^{-uD^2}\| \leq 1$ and $\|e^{-uD_R^2}\| \leq 1$ for $u \geq 0$. By Theorem 3.2 (1) in [LMP], similar to the proof of Lemma 2.2 in [Wa2], then $\|K_1\|_1$ is bounded. By the measure of the boundary of the simplex being zero, we can estimate K_1 in the interior of the simplex, that is $\sigma_0 > 0, \sigma_1 > 0, v_j > 0, v'_j > 0$. We note that the zero order b -pseudodifferential operator is bounded and

$$\|(1 + D^2)^{-\frac{1}{2}} e^{-uD^2}\| \leq L_0 u^{-\frac{1}{2}}; \quad \|e^{-uD^2} - e^{-uD_R^2}\|_1 \leq L'_0 u^r, \quad (4.45)$$

where L_0, L'_0 are constant and r is any integer. It holds that (see line 7 in [BC, P.21])

$$\int_{\Delta_m} v_0^{-\frac{1}{2}} \cdots v_{m-1}^{-\frac{1}{2}} dv = \frac{\pi^{\frac{m}{2}}}{\frac{m}{2} \Gamma(\frac{m+1}{2})}. \quad (4.46)$$

When m is odd, then

$$\frac{\pi^{\frac{m}{2}}}{\frac{m}{2} \Gamma(\frac{m+1}{2})} \leq \frac{2\pi^{\frac{m}{2}}}{(\frac{m+1}{2})!}. \quad (4.47)$$

When m is even, then

$$\frac{\pi^{\frac{m}{2}}}{\frac{m}{2} \Gamma(\frac{m+1}{2})} \leq \frac{2\pi^{\frac{m}{2}}}{(\frac{m}{2})!}. \quad (4.48)$$

So by (4.45)-(4.48), we get

$$\begin{aligned} \|K_1\| &\leq \|C_0\| \|C_1\| \sum_{l,m} \int_{\Delta_1} (\sigma_0)^m (\sigma_1)^l \int_{\Delta_m} \|e^{-\sigma_0 v_0 D^2} - e^{-\sigma_0 v_0 D_R^2}\|_1 \\ &\quad \cdot \|A_{[+]}(1 + D^2)^{-\frac{1}{2}}\| \| (1 + D^2)^{\frac{1}{2}} e^{-\sigma_0 v_1 D^2} \| \\ &\quad \cdots \|A_{[+]}(1 + D^2)^{-\frac{1}{2}}\| \| (1 + D^2)^{\frac{1}{2}} e^{-\sigma_0 v_m D^2} \| dv \\ &\quad \cdot \int_{\Delta_l} \|e^{-\sigma_1 v'_0 D^2}\| \|A_{[+]}(1 + D^2)^{-\frac{1}{2}}\| \| (1 + D^2)^{\frac{1}{2}} e^{-\sigma_1 v'_1 D^2} \| \\ &\quad \cdots \|A_{[+]}(1 + D^2)^{-\frac{1}{2}}\| \| (1 + D^2)^{\frac{1}{2}} e^{-\sigma_1 v'_l D^2} \| dv' \\ &\leq \delta_0 \|C_0\| \|C_1\| \sum_{l,m} \int_{\Delta_1} (\sigma_0)^{\frac{m}{2}} (\sigma_1)^{\frac{l}{2}} \int_{\Delta_m} \int_{\Delta_l} \\ &\quad \cdot \|\delta_1 A_{[+]}(1 + D^2)^{-\frac{1}{2}}\|^{m+l} v_1^{-\frac{1}{2}} \cdots v_m^{-\frac{1}{2}} v'_1^{-\frac{1}{2}} \cdots v'_l^{-\frac{1}{2}} dv dv', \\ &\leq \delta_0 \|C_0\| \|C_1\| \sum_{l,m} \int_{\Delta_m} \int_{\Delta_l} \\ &\quad \cdot \|\delta_1 A_{[+]}(1 + D^2)^{-\frac{1}{2}}\|^{m+l} v_1^{-\frac{1}{2}} \cdots v_m^{-\frac{1}{2}} v'_1^{-\frac{1}{2}} \cdots v'_l^{-\frac{1}{2}} dv dv', \\ &\leq \delta_0 \|C_0\| \|C_1\| \left[\sum_{m, \text{even}} \|\delta_1 A_{[+]}(1 + D^2)^{-\frac{1}{2}}\|^m \frac{2\pi^{\frac{m}{2}}}{(\frac{m+1}{2})!} \right. \\ &\quad \left. + \sum_{m, \text{odd}} \|\delta_1 A_{[+]}(1 + D^2)^{-\frac{1}{2}}\|^m \frac{2\pi^{\frac{m}{2}}}{(\frac{m+1}{2})!} \right]^2 \\ &\leq \delta_0 \|C_0\| \|C_1\| (1 + \delta_2)^2 e^{2(\|\delta_1 A_{[+]}(1 + D^2)^{-\frac{1}{2}}\|)^2} \end{aligned} \quad (4.49)$$

where $\delta_0, \delta_1, \delta_2$ are constant. For the general q , similarly we get

$$\left| \int_{\Delta_q} {}^b\text{StI}^{\text{end}} \left[C_0 e^{-\sigma_0 F} C_1 e^{-\sigma_1 F} \dots C_q e^{-\sigma_q F} \right] d\sigma \right| \leq \delta_0 \frac{1}{q!} (q+1 + \dim X) \left(\prod_{j=0}^q \|C_j\| \right) \left(\delta_1 e^{2\|\delta_1 A_{[+]}(1+D^2)^{-\frac{1}{2}}\|^2} \right)^{q+1}. \quad (4.50)$$

In order to estimate the type II integral, we decompose the type II integral as (4.36). Up to the last term, other terms have the same estimate with corresponding terms in (4.36). Using the same trick as in [Xi], we get that the bound of the 1-norm of the last term is $\delta_0 \frac{1}{q!} (\delta')^{q+1} (q+1) \|B\| \prod_{j=1}^q \|A_j\|$.

By the above estimate, ${}^b\text{ch}(B)$ is well-defined. Similarly, for a fixed $t > 0$, ${}^b\text{ch}(B_t)$ and ${}^b\text{ch}(B_t, \frac{dB_t}{dt})$ are well defined. \square

5 The family Atiyah-Patodi-Singer index theorem for twisted Dirac operators

In this section, we extend the Getzler's index theorem to the family case. Let

$$\widehat{A}(R^{\widehat{N}/X}) = \det^{\frac{1}{2}} \left(\frac{\frac{R^{\widehat{N}/X}}{4\pi\sqrt{-1}}}{\sinh \frac{R^{\widehat{N}/X}}{4\pi\sqrt{-1}}} \right). \quad (5.1)$$

Theorem 5.1 *Suppose that all $D_{M,z}$ are invertible with λ the smallest positive eigenvalue of all $|D_{M,z}|$. We assume that $\|d(p|_M)\| < \lambda$ and $p \in M_{r \times r}(C_{\text{exp}}^\infty(\widehat{N}))$, then in the cohomology of X*

$$\text{ch}[\text{Ind}(pD_{z,+}p)] = \int_{\widehat{N}/X}^b \widehat{A}(R^{\widehat{N}/X}) \text{ch}(\text{Imp}) - \langle \widehat{\eta}^*(B^M), \text{ch}_*(p_M) \rangle. \quad (5.2)$$

Proof. By Theorem 4.6 and $(B+b)(\text{ch}(p)) = 0$, for fixed $t_1, t_2 > 0$, we have in the cohomology of X ,

$$\langle {}^b\text{ch}^*(B_{t_2}), \text{ch}_*(p) \rangle - \langle {}^b\text{ch}^*(B_{t_1}), \text{ch}_*(p) \rangle = -\frac{1}{\sqrt{\pi}} \langle \int_{t_1}^{t_2} \text{ch}^*(B_t^M, \frac{dB_t^M}{dt}) dt, \text{ch}_*(p_M) \rangle. \quad (5.3)$$

Let t_1 go to zero and t_2 go to $+\infty$. By Proposition 5.2 and Theorem 5.3 in [LMP], similar to the computations in Section 4 in [Wa3], we get

$$\lim_{t \rightarrow 0} {}^b\text{ch}^{2k}(B_t)(a_0, a_1, \dots, a_{2k}) = \frac{1}{(2k)!} (2\pi\sqrt{-1})^{-\frac{n}{2}}$$

$$\cdot \int_{\widehat{N}/X}^b a_0 da_1 \wedge \cdots \wedge da_{2k} \widehat{A}(2\pi\sqrt{-1}R^{\widehat{N}/X}). \quad (5.4)$$

Then we have

$$\lim_{t_1 \rightarrow 0} \langle {}^b\text{ch}^*(B_{t_1}), \text{ch}_*(p) \rangle = \int_{\widehat{N}/X}^b \widehat{A}(R^{\widehat{N}/X}) \text{ch}(\text{Imp}). \quad (5.5)$$

By Lemma 5.2 in the following, we have

$$\lim_{t_2 \rightarrow +\infty} \langle {}^b\text{ch}^*(B_{t_2}), \text{ch}_*(p) \rangle = \lim_{t \rightarrow +\infty} {}^b\text{ch}^*(pB_t p). \quad (5.6)$$

By all $D_{M,z}$ being invertible and Proposition 15 in [MP1], we have

$$\text{ch}[\text{Ind}(pD_{z,+}p)] = \lim_{t \rightarrow +\infty} {}^b\text{ch}^*(pB_t p), \quad (5.7)$$

By (5.3) and (5.5)-(5.7) and the definition of the eta cochain form, we get Theorem 5.1. \square

Lemma 5.2 *The formula (5.6) holds.*

Proof. Let $B_{t,u} = \sqrt{t}\psi_t(B + u(2p-1)[B, p])$. Using the same discussions with Theorem 4.6, we get in the cohomology of X

$$\left\langle \frac{\partial {}^b\text{ch}^*(B_{t,u})}{\partial u}, \text{ch}_* p \right\rangle = -\frac{1}{\sqrt{\pi}} \left\langle \text{ch}^*(B_{t,u}^M, \frac{\partial B_{t,u}^M}{\partial u}), \text{ch}_*(p_M) \right\rangle. \quad (5.8)$$

Then

$$\langle {}^b\text{ch}^*(B_{t,1}), \text{ch}_* p \rangle - \langle {}^b\text{ch}^*(B_t), \text{ch}_* p \rangle = -\frac{1}{\sqrt{\pi}} \left\langle \int_0^1 \text{ch}^*(B_{t,u}^M, \frac{\partial B_{t,u}^M}{\partial u}) du, \text{ch}_*(p_M) \right\rangle. \quad (5.9)$$

By $[B_{t,1}, p] = 0$, it holds that

$$\langle {}^b\text{ch}^*(B_{t,1}), \text{ch}_* p \rangle = {}^b\text{ch}(pB_t p). \quad (5.10)$$

By (5.9) and (5.10) and the following lemma, we know that Lemma 5.2 is correct. \square

Lemma 5.3 *The following equality holds*

$$\lim_{t \rightarrow +\infty} \left\langle \int_0^1 \text{ch}^*(B_{t,u}^M, \frac{\partial B_{t,u}^M}{\partial u}) du, \text{ch}_*(p_M) \right\rangle = 0. \quad (5.11)$$

Proof. By $[B_{t,u}^M, p_M] = (1-u)[B_t^M, p_M]$ and $\frac{\partial B_{t,u}^M}{\partial u} = (2p-1)[B_t, p]$, we have

$$\left\langle \int_0^1 \text{ch}^*(B_{t,u}^M, \frac{\partial B_{t,u}^M}{\partial u}) du, \text{ch}_*(p_M) \right\rangle = \sum_{l=0}^{+\infty} \frac{(2l)!}{l!} t^{l+\frac{1}{2}} \sum_{j=0}^{2l} (-1)^{j+l}$$

$$\cdot \int_{u \in [0,1]} (1-u)^{2l} \psi_t \langle p_M - \frac{1}{2}, [B^M, p_M], \dots, (2p_M - 1)[B^M, p_M], \dots, [B^M, p_M] \rangle_{t,u}. \quad (5.12)$$

For the large t , we have

$$\|\mathrm{tr} e^{-tD_u^{M,2}}\| \leq c_0 e^{-t(\lambda-u|dp_M|)^2}. \quad (5.13)$$

In the following, we drop off the index M . Using the same trick in Lemma 4.2 in [Wa3] and (5.13), we get the following estimate. For any $1 \geq \sigma > 0$, $t > 0$ and t is large and any order l fibrewise differential operator A with form coefficients, we have

$$\|e^{-\sigma t B_u^2} A\|_{\sigma^{-1}} \leq C_0 (\sigma t)^{-\frac{l}{2} + \frac{\dim X}{2}} e^{-[(1-\varepsilon)(\lambda-u|dp|)^2 - \varepsilon] \sigma t}, \quad (5.14)$$

where C_0 is a constant and ε is any small positive constant. By (5.14) and the Hölder inequality, we have

$$\begin{aligned} & \left| \langle p - \frac{1}{2}, [B, p], \dots, (2p-1)[B, p], \dots, [B, p] \rangle_{t,u} \right| \\ & \leq C_0 \frac{\|[B, p]\|^{2l+1}}{(2l+1)!} t^{\frac{\dim X}{2}} e^{-[(1-\varepsilon)(\lambda-u|dp|)^2 - \varepsilon] t}. \end{aligned} \quad (5.15)$$

By (5.12) and (5.15), we get

$$\left\langle \int_0^1 \mathrm{ch}^*(B_{t,u}^M, \frac{\partial B_{t,u}^M}{\partial u}) du, \mathrm{ch}_*(p_M) \right\rangle = O(e^{c_0(|dp|-\lambda)t}), \quad (5.16)$$

where c_0 is a positive constant, so Lemma 5.3 holds. \square

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*School of Mathematics and Statistics, Northeast Normal University, Changchun
Jilin, 130024, China*
E-mail: *wangy581@nenu.edu.cn*